CONCERNING THE EXISTENCE OF SURFACES CAPABLE OF CONFORMAL REPRESENTATION UPON THE PLANE IN SUCH A MANNER THAT GEODETIC LINES ARE REPRESENTED BY A PRESCRIBED SYSTEM OF CURVES*

BY

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Introduction.—This paper is in continuation of a previous paper † under nearly the same title. The notation given there is used in this paper with the exception that u, v are here used instead of μ, ν.

We are concerned with a doubly infinite system of *given* curves:

$$f_3(u, v) + Af_2(u, v) + Bf_1(u, v) = 0, \tag{1}$$

of which the differential equation is †

$$a_1 du^3 + a_4 dv^3 + a_2 du^2 dv + a_3 du\, dv^2 + a_5(du\, d^2v - dv\, d^2u) = 0. \tag{2}$$

The geodetic lines of any surface are given by:

$$\begin{aligned}(EF_u - \tfrac{1}{2}EE_v - \tfrac{1}{2}FE_u)du^3 + (-GF_v + \tfrac{1}{2}GG_u + \tfrac{1}{2}FG_v)dv^3 \\ + (EG_u - \tfrac{3}{2}FE_v - \tfrac{1}{2}GE_u + FF_u)du^2 dv \\ + (-GE_v + \tfrac{3}{2}FG_u + \tfrac{1}{2}EG_v - FF_v)du\, dv^2 \\ + (EG - F^2)(du\, d^2v - dv\, d^2u) = 0 \qquad (F_u = \partial F/\partial u, \text{ etc.}),\end{aligned} \tag{3}$$

where u and v are Gaussian coördinates on the surface. A comparison of (2) and (3) leads to the following system of partial differential equations:

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† Transactions of the American Mathematical Society, vol. 2, p. 152.

$$
(4)\quad
\begin{aligned}
&(a) && EF_u - \tfrac{1}{2} EE_v - \tfrac{1}{2} FE_u = \frac{a_1}{a_5}(EG - F^2),\\
&(b) && -GF_v + \tfrac{1}{2} GG_u + \tfrac{1}{2} FG_v = \frac{a_4}{a_5}(EG - F^2),\\
&(c) && EG_u - \tfrac{3}{2} FE_v - \tfrac{1}{2} GE_u + FF_u = \frac{a_2}{a_5}(EG - F^2),\\
&(d) && -GE_v + \tfrac{3}{2} FG_u + \tfrac{1}{2} EG_v - FF_v = \frac{a_3}{a_5}(EG - F^2).
\end{aligned}
$$

The solution of the problem depends upon that of this system.

Multiply equation (a) by $-3F/E$ and add to equation (c). The result is:

$$
(5)\qquad -2FF_u + EG_u - \tfrac{1}{2} GE_u + \tfrac{3}{2}\frac{F^2}{E} E_u = \left(\frac{a_2}{a_5} - 3\frac{a_1}{a_5}\frac{F}{E}\right)(EG - F^2).
$$

Dividing this equation through by $EG - F^2$, integrating with respect to u, and representing by $\psi(v)$ an arbitrary function of v only, we have:

$$
(6)\qquad \frac{EG - F^2}{E^{\frac{3}{2}}} = \psi(v)\, e^{\int \frac{a_2}{a_5} du}\, e^{-3\int \frac{a_1}{a_5}\frac{F}{E} du}.
$$

In like manner, multiplying equation (b) by $-3F/G$, and adding to equation (d), we find

$$
(7)\qquad \frac{G^{\frac{3}{2}}}{EG - F^2} = \phi(u)\, e^{\int \frac{a_3}{a_5} dv}\, e^{-3\int \frac{a_4}{a_5}\frac{F}{G} dv},
$$

where $\phi(u)$ is an arbitrary function of u only.

Beltrami's investigations* were for the case, $a_1 = a_2 = a_3 = a_4 = 0$, that is, where both of the exponentials in the right hand members of (6) and (7) become unity.

In my previous paper I considered the case $F = 0$, that is, where (6) and (7) assume the forms:

$$
(8)\qquad \frac{EG - F^2}{E^{\frac{3}{2}}} = \psi(v)\, e^{\int \frac{a_2}{a_5} du},
$$

$$
(9)\qquad \frac{G^{\frac{3}{2}}}{EG - F^2} = \phi(u)\, e^{\int \frac{a_3}{a_5} dv}.
$$

It is proposed to investigate the case in which $a_1 = a_4 = 0$, while a_2 and a_3 are unrestricted.

Under these restrictions, the system (6) and (7) assumes the form (8) and (9), that is, the same form as for the case $F = 0$. Moreover, these are evidently the only cases in which the right hand members of (6) and (7) are independent of E, G, F.

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*Annali di Matematica, ser. 1, vol. 7 (1866).

§ 1. *Consideration of the form of* (1) *as restricted by the condition that* $a_1 = a_4 = 0$.

Write
$$F_1(u, v) = \frac{f_1(u, v)}{f_3(u, v)}, \quad F_2(u, v) = \frac{f_2(u, v)}{f_3(u, v)}.$$

Then we find:*

$$(10) \qquad a_1 \equiv \frac{\partial F_1}{\partial u}\frac{\partial^2 F_2}{\partial u^2} - \frac{\partial F_2}{\partial u}\frac{\partial^2 F_1}{\partial u^2},$$

$$(11) \qquad a_4 \equiv \frac{\partial F_1}{\partial v}\frac{\partial^2 F_2}{\partial v^2} - \frac{\partial F_2}{\partial v}\frac{\partial^2 F_1}{\partial v^2}.$$

Equating each of these to zero, and integrating, we find:

$$(12) \qquad F_2(u, v) = \psi_1(v) F_1(u, v) + \psi_2(v),$$

$$(13) \qquad F_2(u, v) = \phi_1(u) F_1(u, v) + \phi_2(u),$$

and from these two equations:

$$(14) \qquad F_1(u, v) = \frac{\psi_2(v) - \phi_2(u)}{\phi_1(u) - \psi_1(v)},$$

$$(15) \qquad F_2(u, v) = \frac{\phi_1(u)\psi_2(v) - \phi_2(u)\psi_1(v)}{\phi_1(u) - \psi_1(v)},$$

where $\phi_1(u)$, $\phi_2(u)$, $\psi_1(v)$, $\psi_2(v)$ are as yet arbitrary functions. Hence, when $a_1 = a_4 = 0$, equation (1) assumes the form:

$$(16) \qquad \big(\phi_1(u) - \psi_1(v)\big) + A\big(\phi_1(u)\psi_2(v) - \phi_2(u)\psi_1(v)\big) + B\big(\psi_2(v) - \phi_2(u)\big) = 0.$$

§ 2. *Consideration of the values of the exponentials* $e^{\int\frac{a_2}{a_5}du}$ *and* $e^{\int\frac{a_3}{a_5}dv}$ *when* $a_1 = a_4 = 0$.

We have†

$$(17) \qquad a_2 \equiv 2[F_{1u}F_{2uv} - F_{2u}F_{1uv}] + [F_{1v}F_{2uu} - F_{2v}F_{1uu}],$$

$$(18) \qquad a_3 \equiv 2[F_{1v}F_{2uv} - F_{2v}F_{1uv}] + [F_{1u}F_{2vv} - F_{2u}F_{1vv}],$$

$$(19) \qquad a_5 \equiv F_{1u}F_{2v} - F_{1v}F_{2u} \qquad (F_{iu} = \partial F_i/\partial u, \text{ etc.}),$$

Differentiating (19) with respect to u and comparing the result with (17) we find

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* Previous paper, loc. cit., p. 154.

† Previous paper, loc. cit., p. 154.

(20) $$a_2 = -a_{5u} + 3[F_{1u}F_{2uv} - F_{2u}F_{1uv}],$$

and in like manner, from (18) and (19), we also find

(21) $$a_3 = a_{5v} - 3[F_{2v}F_{1uv} - F_{1v}F_{2uv}].$$

From (12) and (13) we have

(22) $$F_{2u} = \psi_1(v)F_{1u},$$

(23) $$F_{2v} = \phi_1(u)F_{1v},$$

(24) $$F_{2uv} = \psi_1(v)F_{1uv} + \psi_1'(v)F_{1u} = \phi_1(u)F_{1uv} + \phi_1'(u)F_{1v}.$$

These and (20) give

(25) $$a_5 \equiv F_{1u}F_{1v}[\phi_1(u) - \psi_1(v)].$$

Calculating next the bracketed expression in (21), we find it to be equal to

$$F_{1v}\{F_{1uv}[\phi_1(u) - \psi_1(v)] - F_{1u}\psi_1'(v)\} = F_{1v}\frac{\partial\left[F_{1u}\big(\phi_1(u) - \psi_1(v)\big)\right]}{\partial v}.$$

Hence we finally have

(26) $$a_3 = a_{5v} - 3F_{1v}\frac{\partial\left[F_{1u}\big(\phi_1(u) - \psi_1(v)\big)\right]}{\partial v}.$$

Then from (25) and (26) we obtain:

(27) $$\int\frac{a_3}{a_5}dv = \int\frac{\frac{\partial a_5}{\partial v}}{a_5}dv - 3\int\frac{\frac{\partial\left[F_{1u}\big(\phi_1(u) - \psi_1(v)\big)\right]}{\partial v}}{F_{1u}\big(\phi_1(u) - \psi_1(v)\big)}dv$$
$$= \log\frac{F_{1v}}{(F_{1u})^2\big(\phi_1(u) - \psi_1(v)\big)^2}.$$

From (14) we find:

(28) $$F_{1v} = \frac{[\phi_1(u) - \psi_1(v)]\psi_2'(v) + [\psi_2(v) - \phi_2(u)]\psi_1'(v)}{[\phi_1(u) - \psi_1(v)]^2},$$
$$F_{1u} = -\frac{[\phi_1(u) - \psi_1(v)]\phi_2'(u) + [\psi_2(v) - \phi_2(u)]\phi_1'(u)}{[\phi_1(u) - \psi_1(v)]^2},$$

or say, $F_{1v} \equiv A/D^2$, and $F_{1u} \equiv B/D^2$.

Then $\int(a_3/a_5)dv$ becomes $\log(A/B^2)$. By similar reasoning we find that $\int(a_2/a_5)du = \log(-A^2/B)$.

Hence the exponentials $e^{\int\frac{a_2}{a_5}du}$ *and* $e^{\int\frac{a_3}{a_5}dv}$ *are equal, respectively, to* $-A^2/B$ *and* A/B^2*, where* A *and* B *have the meaning given in connection with* (28).

§ 3. *Reduction of the system of partial differential equations* (4) *under the restriction that* $a_1 = a_4 = 0$, *and conditions of integrability for the reduced system.*

Representing the right hand members of (8) and (9) by R and S respectively, and placing p for $(RS)^{\frac{1}{2}}$, we have

$$G = pE, \tag{32}$$

$$F^2 = E(p - RE^{-\frac{1}{2}}). \tag{33}$$

Also write $t = p_u E^{\frac{1}{2}} - R_u$, $h = p_v E^{\frac{1}{2}} - R_v$, $l = 2pE^{\frac{1}{2}} - \frac{3}{2}R$. Transforming system (4), we find for equation (a), after some reductions,

$$Et + (l + R - pE^{\frac{1}{2}})\, E_u - \sqrt{pE - RE^{\frac{1}{2}}}\, E_v = 0. \tag{34}$$

Calculation shows that *equation* (a) *reduces to* (34), and, with a little more difficulty, we find that equations (b) and (d) reduce to the same equation which is:

$$p_v RE - pER_v - pp_u E \sqrt{pE - RE^{\frac{1}{2}}}\, E_u + p(pE^{\frac{1}{2}} - \tfrac{1}{2}R)\, E_v = 0. \tag{35}$$

Hence system (4) *reduces, when* $a_1 = a_4 = 0$, *to equations* (34) and (35). It remains to consider the conditions of integrability for this reduced system.

We find:

$$E_u = 2\frac{E}{R^2} l, \quad E_v = 2\frac{E}{R^2} k, \tag{36}$$

where

$$l \equiv 2\Theta E^{\frac{1}{2}} - RR_u + 2\frac{\Pi}{p}\Delta, \qquad k \equiv 2\Pi E^{\frac{1}{2}} - \frac{\Pi}{p} RR_u + 2\Theta\Delta,$$

in which

$$\Pi \equiv \begin{vmatrix} p & R \\ p_v & R_v \end{vmatrix}, \qquad \Theta \equiv \begin{vmatrix} p & \frac{1}{2}R \\ p_u & R_u \end{vmatrix}, \qquad \Delta \equiv \sqrt{pE - RE^{\frac{1}{2}}}.$$

The condition of integrability, after some reduction, can be written:

$$s^2(kl_E - lk_E) + s(l_v - k_u) + ls_v - ks_u \equiv 0. \tag{37}$$

After a somewhat long calculation we are able to write this in the form:

$$AE^{\frac{3}{2}}\Delta_E + BE\Delta_E + CE^{\frac{1}{2}}\Delta + DE^{\frac{1}{2}} + M\Delta + N + 2\frac{\Pi}{p}\Delta_v - 2\Theta\Delta_u \equiv 0, \tag{38}$$

where

$$A \equiv \frac{8}{R^2}\left(\frac{\Pi^2}{p} - \Theta^2\right), \quad B \equiv 4\frac{R_u}{R}\left(\Theta - \frac{\Pi^2}{p^2}\right), \quad C \equiv \frac{4}{R^2}\left(\Theta^2 - \frac{\Pi^2}{p}\right),$$

$$D \equiv 2\frac{R_u}{R}\left(1 - \frac{\Theta}{p}\right) + 2(\Theta_v - \Pi_u) + \frac{4}{R}(\Pi R_u - \Theta R_v),$$

$$M \equiv 2\left(\frac{\Pi}{p}\right)_v - 2\Theta_u + \Theta\frac{R_u}{R} - \frac{4}{R}\frac{\Pi}{p}R_v,$$

$$N \equiv \frac{\Pi}{p}RR_{uu} + RR_u\left(\frac{\Pi}{p}\right)_u + R_uR_v - RR_{uv} - \frac{\Pi}{p}(R_u)^2.$$

Calculating the values of Δ_E, Δ_u, Δ_v we finally have (38) in the form

(39) $$D_1E - N_1E^{\frac{1}{2}} + (DE^{\frac{1}{2}} + N)\sqrt{pE - RE^{\frac{1}{2}}} \equiv 0,$$

where

$$D_1 \equiv \tfrac{1}{2}pB - \tfrac{1}{4}RA - RC + Mp + \frac{\Pi}{p}p_v - \Theta p_u,$$

$$N_1 \equiv \tfrac{1}{4}RB + RE - \frac{\Pi}{p}R_v + \Theta R_u.$$

Our conclusion is that *the necessary and sufficient conditions that the system of partial differential equations* (34) *and* (35) *shall be integrable are:*

(40) $$D = 0, \quad D_1 = 0, \quad N = 0, \quad N_1 = 0.$$

§ 4. *Consideration of the form which the functions* $\phi(u)$, $\phi_1(u)$, $\phi_2(u)$, $\psi(v)$, $\psi_1(v)$, $\psi_2(v)$ *assume under the conditions of integrability.*

It is desirable to make $\phi(u)$ and $\psi(v)$, which are arbitrary, the means, as far as possible, of satisfying the conditions of integrability. If the third condition, $N = 0$, of (40) were as complicated as the other three, the problem would seem almost beyond our power. Fortunately it is somewhat simpler than the others and by means of the conclusion which we are able to reach from the condition $N = 0$, we are able so to reduce the others that they can be controlled.

Putting $a \equiv -A^2/B$ where A and B have the meaning given in connection with (28), so that

$$P \equiv \frac{a_uA_uA_v - a_uAA_{uv} - a_{uu}AA_v}{a_{uu}A^2}, \qquad Q \equiv \frac{aa_{uv} - a_ua_v}{a^2a_{uu}},$$

we can, after a long reduction, put $N = 0$ in the form:

(41) $$\frac{d\psi(v)}{dv} + 2P\psi(v) - 2Q\psi(v)^{-\frac{1}{2}} = 0,$$

so that

(42) $$\psi(v)^{\frac{3}{2}} = e^{-3\int Pdv}\int e^{3\int Pdv}Qdv + C.$$

Calculation shows that the right hand member of (42) can be a function of N alone, only when $\partial Q/\partial u = 0$ and P is of the form $f_1(u)f_2(v)$. When Q is reduced it assumes the final form:

(43) $$Q \equiv \frac{B(2hB^3 + kA^3)}{A^2N},$$

where

$$h = \phi_1'(u)\phi_2''(u) - \phi_2'(u)\phi_1''(u), \quad k = \psi_1'(v)\psi_2''(v) - \psi_2'(v)\psi_1''(v),$$

and

$$N \equiv A^2B_{uu} - 2AB^2A_{uu} - 2B^2A_u^2 + 4ABA_uB_u - 2A^2B_u^2.$$

Since A is not a factor of B, and each is a function of both u and v (unless we consider a trivial case), it follows that A must be a factor of $2hB^3 + kA^3$. This condition is satisfied, with least restriction upon the form of the curve, by putting $h = 0$. Then as a further condition, either the product of AB and a function of v only must be equal to N, or else k must vanish, and we find that the latter includes the former. Hence Q must vanish.

The first restriction is that $\phi_1(u)'\phi_2(u)'' - \phi_2(u)'\phi_1''(u)$ *and* $\psi_1'(v)\psi_2''(v) - \psi_1'(v)\psi_2''(v)$ *each vanish.* Integrating, we have

(44) $$\begin{aligned} \phi_2(u) &= \phi_1(u) + k_1u + k_2, \\ \psi_2(v) &= \psi_1(v) + h_1v + h_2, \end{aligned}$$

where h_i and k_i are to be determined. Integrating the equation $Q = 0$ with respect to v, we find, after some reduction, that the following expression must be a function of u only:

$$\frac{m_1 + m_2\psi(v)\psi'(v) + m_3v\psi_1(v) + m_4\psi_1(v) + m_5\psi_1'(v) + m_6v}{n_1 + n_2\psi(v)\psi'(v) + n_3v\psi_1(v)\psi_1'(v) + n_4v\psi(v) + h_1\phi_1'(u)v^2\psi_1''(v)},$$

where m_i and n_i do not contain v. Calculation shows that h_1 and k_1 must vanish, after which the above expression assumes the form $-2\phi_1''(u)/l\phi_1'(u)$, where $l = -k_2 + h_2$.

Calculating now the value of P, we find it to be $-2\psi_1''(v)/\psi_1'(v)$, so that

$$\frac{d\psi(v)}{dv} - 2\frac{\psi_1''(v)}{\psi_1'(v)}\psi(v) = 0,$$

or

$$\psi(v) = k[\psi_1'(v)]^2.$$

Collecting our results thus far, we have

(45) $$\begin{aligned} \phi_2(u) &= \phi_1(u) + k_2, \\ \psi_2(v) &= \psi_1(v) + h_2, \\ \psi(v) &= k[\psi_1'(v)]^2. \end{aligned}$$

It remains to consider the remaining three conditions of integrability. Of these, $N_1 = 0$ is the most simple. A little consideration shows that it must be of the form $\phi''(u) + \beta_2\phi'(u)/\beta_1 + \beta_3/\beta_1 = 0$, where β_i is independent of $\phi(u)$. It turns out that $\beta_2 \equiv 0$, and that β_3/β_1 is equal to

$$(46) \qquad -\lambda k^{\frac{1}{3}} \frac{[\psi_1''(v)]^2}{\psi_1(v)} \phi_1''(u) - \left\{ \frac{17}{k} \frac{[\psi_1''(v)]^2}{[\psi_1'(v)]^6} - \frac{2}{k} \frac{\psi_1'''(v)}{[\psi_1'(v)]^5} \right\} [\phi_1'(u)]^2.$$

This must be a function of u only, which requires that $\psi_1(v)$ satisfy the system:

$$(47) \qquad \begin{aligned} [\psi_1''(v)]^2 &= \lambda_1 \psi_1'(v), \\ 17\,[\psi_1''(v)]^2 - 2\psi_1'''(v)\psi_1'(v) &= \lambda_2 [\psi_1'(v)]^6, \end{aligned}$$

where λ_i is constant. The system (47) admits of but one solution, $\lambda_i = 0$, $\psi_1''(v) = 0$. That is, $N_1 = 0$ adds to (45) the two results:

$$(48) \qquad \begin{aligned} \phi(u) &= b_1 u + b_2, \\ \psi_1(v) &= a_1 v + a_2. \end{aligned}$$

The conditions, $D = 0$, and $D_1 = 0$, remain for consideration. The equations (45) and (48) reduce $D = 0$ to the form:

$$pR_u - p(R_u)^2 + \tfrac{1}{2} R R_u p_u = 0,$$

which is satisfied by either $R_u = 0$, or $p - pR_u + \frac{1}{2} Rp_u = 0$. A little calculation shows that $D_1 = 0$ is also satisfied if $R_u = 0$. It remains to determine whether $D_1 = 0$ is also satisfied when $\Theta = p$. We find that $D_1 = 0$ assumes the form:

$$pR_u - Rp_u = \tfrac{2}{3} p,$$

which we are to consider simultaneously with $pR_u - \frac{1}{2} Rp_u = p$. This requires either that $p = 0$, or that $R_u = \frac{4}{3}$.

If $p = 0$, then, either $G = 0$, or $EG - F^2 = 0$, the latter of which is excluded. If $G = 0$ we find that $E = 1/R^2$ and $F = \pm 1$, and the corresponding surface is an imaginary ruled surface.

The assumption $R_u = 0$ would lead us to the relation

$$(b_1 u + b_2)^2 (\tfrac{4}{3} u + \delta)^3 = \lambda^4 k^2 a_1^{12} c^2,$$

which would require that $\phi(u)$ vanish, and hence lead us to the same imaginary surface as for $p = 0$.

Hence: *In order that a real surface exist, it is necessary and sufficient that we have:*

$$(49) \qquad \begin{aligned} \phi(u) &= b_1 u + b_2, & \psi(v) &= k a_1^2, \\ \psi_1(v) &= a_1 v + a_2, & \phi_1(u) &= c_1 u + c_2, \\ \psi_2(v) &= \psi_1(v) + h_2, & \phi_2(u) &= \phi_1(u) + k_2, \end{aligned}$$

where the constants must not cause $\phi(u)$ or $\psi(v)$ to vanish.

If now we calculate a_2 and a_3 we find that they also vanish, so that we are led to the conclusion that *there is for our proposed problem no new solution.*

§ 5. *Integration of the system of partial differential equations.*

The conditions of integrability (40) being satisfied, we proceed to integrate the corresponding systems, (34) and (35), of partial equations.

They may now be written:

$$(50) \qquad E_u = -2nE^{\frac{3}{2}}, \quad E_v = -2nE^{\frac{3}{2}}\sqrt{p - RE^{-\frac{1}{2}}},$$

where n is the constant $b_1/(k_2 - h_2)k^{\frac{1}{2}}a_1c_1$.

The integrals of these are, respectively,

$$(51) \qquad E^{-\frac{1}{2}} = nu + f_1(v), \quad \sqrt{p - RE^{-\frac{1}{2}}} = -\tfrac{1}{2}nv + f_2(u),$$

where $f_1(v)$ and $f_2(u)$ are yet to be determined.

We have the identical relation:

$$p - [f_2(u) - \tfrac{1}{2}nv]^2 \equiv R\,[nu + f_1(v)].$$

Writing $u = 0$, we have at once:

$$f_1(v) \equiv \frac{p_0 - [f_2(0) - \frac{1}{2}nv]^2}{R},$$

and further consideration shows that $f_2(u) \equiv f_2(0)$, a constant which we denote by δ. Writing $D \equiv m(b_1u + b_2) - [\delta - \frac{1}{2}nv]^2$, we finally have:

$$(52) \qquad E = \frac{R^2}{D^2}, \quad F = \frac{R(\delta - \frac{1}{2}nv)}{D}, \quad G = \frac{m(b_1u + b_2)R^2}{D^2}.$$

§ 6. *Curvature of the surface corresponding to the problem considered.*

The Gaussian curvature of any surface is given by:

$$K = \frac{1}{\sqrt{EG - F^2}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{EG - F^2}}{E}\begin{Bmatrix}11\\2\end{Bmatrix}\right) - \frac{\partial}{\partial u}\left(\frac{\sqrt{EG - F^2}}{G}\begin{Bmatrix}12\\2\end{Bmatrix}\right)\right],$$

where,

$$\begin{Bmatrix}11\\2\end{Bmatrix} \equiv \frac{-FE_u + 2EF_u - EE_v}{2(EG - F^2)}, \quad \begin{Bmatrix}12\\2\end{Bmatrix} = \frac{EG_u - FE_v}{2(EG - F^2)}.$$

Referring to system (4) we find that $\begin{Bmatrix}11\\2\end{Bmatrix} = \frac{a_1}{a_5}$. Writing $EG - F^2 = \Delta$, we have, from the same system,

$$\frac{EG_u - FE_v}{\Delta} = \frac{a_2}{a_5} + \tfrac{1}{2}\,\frac{GE_u}{\Delta} + \frac{FE_v - FF_u}{\Delta},$$

$$-\frac{F}{E}\frac{a_1}{a_5}=\tfrac{1}{2}\,\frac{FE_v-FF_u}{\Delta}+\tfrac{1}{2}\,\frac{F^2}{E}\frac{E_u}{\Delta},$$

and from these,

$$\left\{\begin{matrix}12\\2\end{matrix}\right\}=\tfrac{1}{2}\frac{a_2}{a_5}-\tfrac{1}{2}\frac{F}{E}\frac{a_1}{a_5}+\tfrac{1}{4}\frac{E_u}{E}.$$

Put

$$M=\frac{G^{\frac{3}{4}}}{\phi(u)^{\frac{1}{2}}E},\quad N=\frac{\psi^{\frac{1}{2}}}{E^{\frac{1}{4}}},$$

$$a=-\tfrac{1}{2}\frac{a_3}{a_5}+\tfrac{3}{2}\frac{a_4}{a_5}\frac{F}{G},\quad \beta=\tfrac{1}{2}\frac{a_2}{a_5}-\tfrac{3}{2}\frac{a_1}{a_5}\frac{F}{E},$$

$$H=\frac{\sqrt{\Delta}}{E},\quad \left\{\begin{matrix}11\\2\end{matrix}\right\}=W,\quad \left\{\begin{matrix}12\\2\end{matrix}\right\}=U.$$

Then from (6) and (7), we have

$$H=Me^{\int a\,dv}=Ne^{\int\beta\,du},$$

and some calculation gives:

$$K=\frac{1}{E}\left\{\frac{\partial}{\partial v}\left(\frac{a_1}{a_5}\right)-U_u+\begin{vmatrix}\dfrac{a_1}{a_5} & \beta+\dfrac{\partial\log N}{\partial u}\\ U & a+\dfrac{\partial\log M}{\partial N}\end{vmatrix}\right\}.$$

Then calculating U_u, $\partial\log N/\partial u$, and $\partial\log M/\partial v$ we find as a final expression for *the total curvature of a surface corresponding to the system of partial differential equations* (4):

$$K=\frac{1}{E}\left\{\frac{\partial}{\partial v}\left(\frac{a_1}{a_5}\right)-\tfrac{1}{2}\frac{\partial}{\partial u}\left(\frac{a_2}{a_5}\right)-\tfrac{1}{4}\frac{\partial^2\log E}{\partial u_2}+\frac{F}{E}\frac{\partial}{\partial u}\left(\frac{a_1}{a_5}\right)+\tfrac{1}{2}\frac{a_1}{a_5}\frac{EF_u-FE_u}{E^2}\right\}$$

(53)

$$+\frac{1}{E}\begin{vmatrix}\dfrac{a_1}{a_5} & \tfrac{1}{2}\dfrac{a_2}{a_5}-\tfrac{3}{2}\dfrac{a_1}{a_5}\dfrac{F}{E}-\tfrac{1}{4}\dfrac{E_u}{E}\\ \tfrac{1}{2}\dfrac{a_2}{a_5}-\tfrac{1}{2}\dfrac{a_1}{a_5}\dfrac{F}{E}+\tfrac{1}{4}\dfrac{E_u}{E} & -\tfrac{1}{2}\dfrac{a_3}{a_5}+\tfrac{3}{2}\dfrac{a_4}{a_5}\dfrac{F}{G}+\tfrac{3}{4}\dfrac{G_v}{G}-\dfrac{E_v}{E}\end{vmatrix}.$$

If $a_1=a_4=0$, this reduces to:

$$K=\frac{1}{E}\left\{-\tfrac{1}{2}\frac{\partial}{\partial u}\left(\frac{a_2}{a_5}\right)-\tfrac{1}{4}\frac{\partial^2\log E}{\partial u^2}-\tfrac{1}{4}\left(\frac{a_2}{a_5}\right)^2+\tfrac{1}{16}\left(\frac{E_u}{E}\right)^2\right\}.$$

But then, $E=R^2/(A-B^2)^2$, where $A=mb_1u+mb^2$ and $B=\delta-\tfrac{1}{2}nv$.

Hence:

$$\frac{1}{16}\left(\frac{E_u}{E}\right)^2 - \frac{1}{4}\frac{\partial^2 \log E}{\partial u^2} = -\frac{1}{4}\frac{m^2 b_1^2}{(A - B^2)^2}.$$

Making use of this relation we have:

$$K = \frac{1}{E}\left[-\frac{1}{2}\frac{\partial}{\partial u}\left(\frac{a_2}{a_5}\right) - \frac{1}{4}\left(\frac{a_2}{a_5}\right)^2\right] - \frac{1}{4}\frac{m^2 b_1^2}{R^2}.$$

Here we notice that if a_2 were zero the curvature would be constant—which agrees with known results.

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